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Two Reflexivities

Scholastic and Cartesian Second Intentions in Klein's *Greek Mathematical Thought and the Origin of Algebra*

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"I don't think the Greeks had concepts.
Concepts have cogito-like reflexivity."
Ed Halper[2]

Introduction

Jacob Klein has several characterizations of the distinctive manner in which algebraic symbols exist. I have found eight:

1. The objects represented by, for example, numerical variables and constants, are identified with the means of their representation, for example, *x*, *y*, *a*, *b*, =, and signs for operations (123, 178, 224). Accordingly, a letter sign (or Cartesian line segment) has meaning only within the sign system as a whole with its constitutive rules of calculation; it is "comprehensible only within the language *of symbolic formalism*" (174–76 [here 175], see also 193).
2. Potential determinacy is taken as actual determinacy in calculation, that is, we calculate (add, multiply, subtract, divide) with indeterminate *x* and *y* as if they were determinate numbers (123, 175).

3. The mode of being of the object, for example, a multitude of units, is identified with the mode of being of the general concept related to the object, for example, "multitudinousness as such," "mere multiplicity in general" (192, 200–2, 208; *WPNW* 26).
4. The object of a second intention is represented by and as the object of a first intention, namely, a particular letter sign or Cartesian line segment (174–75, 192, 207–8; *WPNW* 26; *MR* 60–63).
5. Continuous variability of ratio-numerical magnitudes (letter sign or line segment): a second intention is taken as first intention via Kantian schematic imageability (*WPNW* 20–21, 26).
6. Dimensionlessness: numbers are conceived as pure (dimensionless) ratios with the unit as the consequent (218–23).

Overarching and emblematic of all of the modern conceptions (Viète, Stevin, Descartes, Wallis):

7. Rejection of the distinction between discrete number (*arithmos*) and continuous magnitude (*megethos*) (178, 219–21).
8. Rejection of the tenet, one is not a number but the principle of number (191, 212).

I focus here on the fourth: Klein's use of the scholastic doctrine of first and second intentions to clarify the symbolic concept of number in Part II of *GMTOA*. Note, however, the importance of *calculation* in the first two descriptions, and thus of the syntactical rules (the axioms) of calculational operations ("These rules . . . create the systematic context which originally 'defines' the object to which they apply . . . namely an object of 'calculational' operations"—176; *GL*-II 184).[3] We shall see that the fourth description would be impossible without calculational syntax (the axioms). We shall also see that the fourth description would be impossible without a distinctively Cartesian difference from the scholastic account of second intentions. The difference concerns *reflexivity*—scholastic versus Cartesian—and is presented by Klein on page 208 of *GMTOA*. The purpose of this paper is to clarify that account.

Klein's most complete presentation of the scholastic doctrine of second intentions as a means to understanding the conceptual structure of symbols is in Chapter 12B of *GMTOA*, on Descartes.[4] Klein there uses Eustachius of Sancto Paulo (1573–1640) as a source for the scholastic doctrine. In the present paper I supplement Klein with Joseph Owens's account of intentional beings, first and second, in his *Elementary Christian Metaphysics*. Owens is useful for enabling us to see the difference between scholastic second intentions and Cartesian second intentions because Klein is not completely clear at *GMTOA* 208 as to where the scholastic account ends and the Cartesian account (as interpreted by Klein) begins. It is the latter (Cartesian) that leads to symbols and symbolic mathematics, not the former (scholastic).[5] As noted, the difference involves reflexivity, and, ultimately, the relation between mind and world as understood by scholastic-Aristotelianism (shown below to be a form-receptive realism[6]) and as conceived by Descartes (as explicated in the following).

Among the modern founders examined by Klein, Descartes is unique in being the most philosophical, thus most aware of the need to give an account of his own doings. In the *Rules for the Direction of the Mind*, Descartes endeavors, "with an explicitness perhaps novel in the history of science," to explain the "new mode of 'abstraction' and a new possibility of 'understanding'," namely, through the language of algebraic equations and their graphs (200). The basis of Descartes's epistemology of symbolic abstraction (Klein: *symbolische Abstraktion*)[7] is his model of pure intellect and corporeal imagination, the latter being "a true part of the body," "nothing but a real body with real extension and shape,"[8] in Rules 12 through 14. There, intellect, pure of all images, "applies itself"[9] in a twofold way to the corporeal imagination: first, by *passively* receiving figures or traces imprinted on the imagination (via nerves and animal spirits from sense organs and the external objects that stimulate them), then, second, by *actively* using the imagination to form new figures. But new figures of what?

Between the passive and active uses of imagination, pure intellect exercises a power proper to it alone, namely, that of "separating abstract entities,"[10] such as "mere multitude" (*solam multitudinem*)[11] or (as Klein calls it) "multitudinousness as such" (200, 201; *Mengenhaftigkeit als solche—GL*-II 209, 211),[12] that is, generic manyness, which, like anything generic (say, generic animal or generic triangle), is unimaginable (it lacks individuality and specificity) and exists only in the intellect. But this—generic manyness—is what is *symbolized* by figure-forming, intellect-active imagination: "*Thus the imaginative power makes possible a symbolic representation of the indeterminate content which has been 'separated' by the 'naked' intellect*" (202, emphasis Klein's).[13]

It will be argued in the following that, whatever its precise meaning, "multitudinousness as such" (*sola multitudo*) does not refer to discrete number (*arithmos*) in contradistinction to continuous magnitude (*megethos*). Rather, in accordance with description 7, above, it supersedes that long-standing distinction. In accordance with description 8, it rejects the long-standing tenet that *one is not a number but the principle of number* (the arithmetical unit or monad is not a principle of multitudinousness as such).

The algebraic symbol, x, intends generic manyness, like a concept, yet is a sense-perceptible individual that is subject to calculational operations (see note 3). *Calculability* seems to be the determinate sense of the indeterminate x. This is striking, even paradoxical. It is as if there were "generic animal," which can exist only in the intellect, that we could nevertheless feed and pet.[14] As Hopkins aptly puts it:

> The indeterminate [it means neither multitudes of units nor geometrical magnitudes; *OLSM* 518] general object yielded in "symbolic abstraction" is neither purely a concept nor

> purely a "sign," but precisely the unimaginable and unintelligible identification of the object of a second intention with the object of a first. This identification is "unimaginable" because "images" properly . . . refer either to particular objects of first intentions or to their particular "common qualities." The identification of second and first intentional objects is "unintelligible" because, for "natural" predication, to say that a concept is both general and particular "at the same time" is nonsensical. (*OLSM* 509–10)

The distinction within ordinary predication between first and second intentions is elaborated by means of Owens in Part II–IV, following. To introduce and motivate that account, consider the two sentences, "Socrates is a man" and "man is a species." The term *man* cannot mean the same thing in these two sentences lest it follow that Socrates is a species. There are two different concepts of man here, or, more generally, two types of concepts differing in how they are formed and how they refer to their objects. 'Man' as understood in the first sentence is called a first intention. 'Man' as known in the second sentence is called a second intention.

The once-novel products of intellect and imagination pioneered by Descartes (as well as Viète, Stevin, and Wallis) became over three centuries the symbols—letter signs and numerals, equations, coordinates and graphs—of our now taken-for-granted algebra, analytic geometry, and advanced mathematics.[15]

Klein's account of Descartes addresses two interconnected questions: (1) How does Descartes's model of pure intellect and corporeal imagination work? (2) What is the distinctive conceptual structure of the resulting algebraic symbols and symbolic geometrical figures? In addressing these questions, it is essential to keep the following remark in mind:

> Here the constant presupposition is that *the "pure" intellect in itself has no relation at all to the being of the world*

> *and the things in the world*. What characterizes it is not so much its "incorporeality" as just this unrelatedness. (202; see also *MR* 63)

Descartes conceives mind and world as radically separate.[16] This is the key difference between Cartesian and scholastic intentionality. It means that in the Cartesian theory of knowledge there is nothing like the scholastic first intention. This is at present not obvious; Owens's account of scholastic intentionality, below, will enable us to see it.

For Cartesian mind to come into contact with the world, the passive and active (figure-forming) functions of the Cartesian imagination are required, especially the latter. Let us proceed with Klein's account in *GMTOA*, Chapter 12B. (The other approach would be to analyze Descartes's algebra of line segments set out in the opening pages of his *Geometry*, for which see Romiti, *CM*, Chapter 3 or "The Symbolic Space of Descartes's *Geometry* and its Symbolic Mathematical Underpinnings," contained in the present volume.)

I. Klein's Use of the Scholastic Intentionality Doctrine—Part I: Eustachius

It is worth quoting Klein's introduction of the scholastic doctrine at length. In the following, brackets enclose supplementary references and an explanatory phrase. Klein's original German of the most important clause is inserted in parentheses:

> We are now able, by using Descartes' assertions as a basis and taking into account the contemporary literature of the schools, to fix yet more exactly that conceptual character of algebraic symbols which has already been variously outlined [174, 184]. We saw that Descartes designates the "sola multitudo" which the intellect "separates" from the "idea"

> of number [for example, an array of points, 201 (see also *MR* 62)] it finds available in the imagination as an "abstract being" (ens abstractum), also called an *ens rationis* in the language of the schools. In the *Summa* of Eustachius a Sancto Paulo, IV, 17–19 (quoted after Gilson, *Index*, p. 107) three kinds of "entia rationis" are enumerated: "Beings of reason are either negations, or privations, or *second intentions*. . . . The two first kinds appertain to things in their own mode before any operations of the intellect." . . . [O]nly the third kind owes its "being" to the operation of the intellect alone: "But the last kind does not belong to things *unless a certain operation of the intellect is presupposed*, wherefore these beings of reason are said to depend on the intellect for their existence and connection; . . . this is why a 'being of reason' in its proper and strict sense is agreed to be only the last kind." . . . Thus "mere manyness" (sola multitudo), multitudinousness as such, which has its "being" by grace of the "pure intellect," is truly an *ens abstractum* or *ens rationis* in the sense of a "second intention." . . . Now Eustachius, appealing to established usage, defines "second intention" more narrowly as an *ens rationis* "which is conceived as belonging to a thing known by virtue of its being known, and which cannot exist except *objectively in the intellect*, since it is conceived [not originally but] *secondarily and by a reflexive operation of the mind*" (*und das in keinem anderen Sinne 'sein' kann, als* für den Verstand gegenständlich ist: *denn es wird ja [nicht ursprünglich, sondern] nachträglich und durch ein* auf sich selbst bezogenes Tun des Verstandes *erfasst*) (quodque non aliter potest existere quam objective in intellectu, cum secondario et per reflexam mentis operationem concipitur). (206–7; *GL*-II 221)[17]

Here we are given Eustachius on second intentions and *reflexivity*, after Descartes on multitudinousness as such, and a particular multitude, say, five points, on the corporeal imagination. Reflexivity is the ability of the mind to "look" in different directions and to turn from looking wholly "outward" at extra-mental beings (not at all at itself) to looking, partly or wholly, at itself in its own activity and products (it is "a self-related operation of the mind"—see *GL*-II 221). Klein quotes Descartes, *Meditations*: "'the mind, when it thinks [reflexively], in a way turns *itself toward itself*' (. . . mens, dum intelligit, se ad seipsam quodammodo convertat; cf. *Meditations* VI, [AT VII 73:15 ff.])" (200). Similarly, Hopkins speaks of "Vieta's redirection of his cognitive regard" (*OLSM* 521). Owens, as discussed below, speaks of "the intellect in its reflexive [not direct] gaze" (*ECM* 239).

Note that Klein, using Eustachius, has proceeded to second intentions, not by prior contrast with first intentions,[18] but from within the category of beings of reason. The alternative order of presentation would be, within the scholastic doctrine of intentional beings, to set out first intentions, then second intentions.[19] Owens is useful because he gives an account of first intentions, then second intentions, in terms of the turning "gaze" (an activity) of the intellect, or reflexivity. From Owens, below, we shall learn that man and animal are first intentions; species and genus are second intentions. We thus have at this point a fruitful question: why is mere manyness necessarily a *second* intention? Why could it not be a first intention related to a particular multitude, say, five units, as man and animal are related to a particular human being, say Socrates? Granted, a substance is precisely one and not many.[20] But could we not make an analogy—despite disanalogies—between one substance (Socrates) and one multitude (five units), predicating man and animal of the former and fiveness (like man) and mere manyness (like animal) of the latter? This question gets to the heart of the matter. I have asserted above that mere manyness (*sola multitudo*) does not refer to discrete multitude in contradistinction to continuous magnitude, that it falls under descriptions 7 and 8 of modern symbolic number. Does the proposed analogy (between a substance and a number of units) likewise overcome the discrete-continuous distinction, or rather does it leave it intact? If the latter, it would not get us to Descartes's *sola multitudo*. I attempt to resolve this in Part V, below. For now let us turn to Owens's account. I should note that I find Klein, Eustachius as reported by Klein, and Owens all in accord on the scholastic doctrine.

II. Owens: First Intentions

A preliminary caution: Although useful, Owens's description is potentially confusing for us because he employs the scholastic and pre-Cartesian meaning of *subject* and *object*. For example, "[t]he one real thing becomes a number of different objects" (*ECM*, 238). In this now-forgotten understanding, it is the subject that is outside the intellect; the object is in the intellect. For example, Eustachius, quoted above: "*[quod] non aliter potest existere quam objective in intellectu*" ("which cannot exist except objectively in the intellect"), rendered by Klein (*GL*-II 221) as "*in keinem anderen Sinne 'sein' kann, als für den Verstand gegenständlich ist*" ("can exist in no other sense but objectively for the intellect"). So the fire in the fireplace—by which I can warm or burn myself—is the subject; the fire in my knowledge is the object (literally "thrown against" my intellect by the subject). I cannot warm or burn myself by the objective fire. In modern, post-Cartesian language, I am the subject, the fire in my mind is the concept, the fire in the fireplace is the object. This suggests, however, that the Cartesian object could be identified with the scholastic subject. Anyone familiar with Descartes's science knows how mistaken this identification would be. To continue with the example above, for scholastic-Aristotelian realism, the fire is in itself hot and luminous; for Cartesian science, heat and light are intramental re-presentations of purely corpuscular interactions—thus, a concept-construction or model replaces a form-matter substance for a mind that is conceived to have no direct access to the world.[21] The question of the distinction between what first exists outside the intellect—the things to be known—and their subsequent existence as known by the intellect is removed (this is implied by description 3, above). For Klein, "[t]his means that the one immense difficulty within ancient ontology, namely, to determine the relation between the 'being' of the [extramental] object itself and the 'being' of the object in thought [intentional being], is . . . accorded a 'matter-of-course' solution whose presuppositions and the extent of whose significance are simply bypassed in the discussion" (192).[22] Symbolic concept-formation, and thus modern algebra, is a necessary condition of this historic transformation in mind-world relation, and thus in our understanding of what things are (see *OLSM* 4).[23]

Back to the task at hand: In the following scholastic account by Owens, the object always exists in the intellect. Intentional being, whether first or second, is always cognitional or *objective* being, being as known. Outside of the intellect are *real* beings—determinate and sensible individuals possessing *per se* natures or quiddities in function of their form-matter composition.[24] Accordingly, for the reading of Owens, I designate *object* in this sense either as "[scholastic] object," in quoting Owens, or as object_s with subscript *s*. My intention is to bring into clear relief the transition to Cartesian (wholly self-relating) second intentions, and thus finally to symbols on *GMTOA* 208.

Owens:

> Socrates is known as Socrates, as a man, as an animal, as a living thing . . . in a process by which he is seen respectively in comparison with other men, other animals, other living things. . . . All these aspects [he's a man, an animal, alive] . . . are present in the sensible individual as first known [that is, the contents known—humanity, animality, life—are not constituted by the operations of the intellect, are not beings of reason]. . . . The wider aspects are

> isolated through the process of [Thomistic] abstraction, by successively leaving the differentiae out of consideration. In all these abstractions the sensible thing is what is *directly* known, as Socrates, a man, an animal, and so on. The gaze of the intellect is still focused upon the sensible thing itself. The *direct* gaze of the intellect upon the thing itself throughout these various abstractions is called technically the first intention. (*ECM* 237–38; italics and bracketed explications mine)[25]

I have italicized *direct* because it is about to be contrasted with *reflexive* in the transition from first to second intentions. Let us prepare that by the following considerations: "All these aspects . . . are present in the sensible individual as first known." There is in Socrates the quiddity or nature *man*, the quiddity or aspect *animal*.[26] The universal *man* is a first-intentional object_s or object_s of a first intention. Similarly, the universal *animal* is a first-intentional object_s. In contrast, the real being, the sensible individual existing independently of the intellect of the knower (the scholastic subject), is Socrates.

Note the sharp contrast with Cartesian mind-world separation (discussed above): Owens's judgment that man and animal "are present *in* the sensible individual" (after *Physics* 192b22; my emphasis) as real, Aristotelian and Thomistic natures that are then received in the soul is, for Descartes, mistaken. Since, for Descartes, no such natures are received, it follows that, in the Cartesian theory of knowledge, there is nothing like the scholastic first intention; see note 21.

Owens speaks here of first intentions in scholastic *natural* philosophy, whose object_ss (like "snub"—see *Physics*, 194a7) depend on sensible matter both to be and to be understood. Our interest is not in natural philosophy but in mathematics, whose object_ss may or may not first exist in sensible matter (Aristotle says they do, Plato says they do not) but, in any case, do not depend on sensible matter to be understood. What can be said, in preparation for Klein on Descartes, about mathematicals?

Klein shows that the Greek *arithmos* always meant a definite amount of definite things. For Greek thought, an *arithmos* is particular (not general) and determinate (not indeterminate), for example, four monads, represented as :: (four dots) or as $M^o\delta$ (in Diophantine notation). Therefore, an *arithmos* as known, as existing in the intellect of the mathematician, is a first intention (more precisely, an object_s of a first intending).[27] To determine the amount in an *arithmos*, we start the count with *one*, for example, one apple, or one pure arithmetical unit (*monad*), because it is the first of the definite things (apples or pure units) present. The unit, without which the count could not begin, is always, for Greek thought, a positive being, whether sensible (an apple) or intelligible (an arithmetical unit). Therefore, as known, as existing in the intellect, the unit is an object_s of a first intention (first intending). What about Greek geometry? The drawn figures visible to us in Euclid and Apollonius are imperfect images of perfect (ideal) forms intelligible to us (somehow). Both the visible figure and, more importantly, the intelligible ideal form (the circle or the ellipse) are, according to Klein, particular—not general like the graph of a circle or ellipse in Cartesian coordinates—and determinate in contradistinction to the variables (x, y) and constants (r, a, b) in the equations of the circle ($x^2 + y^2 = r^2$) and ellipse $(x/a)^2 + (y/b)^2 = 1$ (a difference of possible vs. actual determinacy, as in description 2 above). "Apollonius has in view the *particular* ellipse. . . . The representation in the drawing gives a true '*image*' [*Abbild*] of . . . *this* ellipse" (*WPNW* 16–17). Thus Greek geometry was also first-intentional. Accordingly (as in natural philosophy), the content known in Greek arithmetic and geometry is understood to derive from the intrinsic properties of the things known; it is not constituted by the operations of intellect as is a second intention or being of reason (207–8).

The important case of the general theory of proportions (based in Book V of Euclid's *Elements*), which demonstrates theorems, for example, alternation, for both discrete number (VII.13) and continuous magnitudes (V.16), is discussed briefly in note 37, below. The essential question is, did that general theory posit a correspondingly general (thus univocal) object, like our real number, *x*? I believe the answer is no (for example, alternation of magnitudes is still subject to homogeneity requirements; see note 37).

III. Transition to Owens on Second Intentions

Is there in Socrates the quiddity or nature *species*, the quiddity or common nature *genus*? If so, then we could say, "Socrates is a man, Socrates is a species, Socrates is an animal, Socrates is a genus." But we cannot. The predications, "Socrates is a man," "Socrates is an animal," "man is a species," and "animal is a genus" fall into two different classes with respect to the relation between knower and known: first-intentional and second-intentional. As mentioned above, there are two different concepts of man here or, more generally, two types of concepts differing in how they are formed and how they refer to their objects. In the predications "man is a species" and "animal is a genus" the content known—*species*, *genus*—does not have prior existence in the real individual being. Humanity and animality do (in the scholastic account) have prior existence in the real individual being. Where do the meanings of *species*, *genus*, etc., come from? Answer: from the comparing, separating, grouping operations of intellect, which it performs *after reflecting* on what it has known directly. This means that the meanings of *genus* and *species* here, in the second intention, are partially constituted by the cognitive operations of the intellect, but not wholly, because those meanings are *founded on* the aspects (to paraphrase Owens) known directly, in the first intention. Thus, we could say that the intellect is now looking partly at itself in its own cognitional processes, and partly out at the world of natural kinds (the source of its data—see *De Anima* 3.8, 431b20ff.).

IV. Owens: Second Intentions

Owens, again:

> The intellect is also able to reflect on its own activities and processes. It is able to see that what is one in real being has become multiple in cognitional being, as the one real man is represented separately as Socrates, as man, as animal, as living thing, as body, and as substance. The same thing is seen as several [scholastic] objects, each object having its own separate act of intentional being. These objects of simple apprehension are common natures. Each just in itself has no being, but is able to be either in reality or in cognition. Each is seen by the intellect, first in its real existence, and then, through reflection, in its intentional being. Considered separately as it is in intentional being, each appears as a representation through which the real thing is known.
>
> As the intellect in its reflexive gaze views each of these representations separately, it sees them as ever widening objectivations of the same real thing. It compares them with one another. It sees that the [scholastic] object "man" leaves out of consideration the individual characteristics of Socrates, and so can be found identified equally well in reality with Plato, Caesar, Kennedy, Castro, and innumerable other instances. Understood in this way, the [scholastic] object "man" is technically called a lowest universal or a *species*. The predication "Man is a species" may be made. From the same viewpoint, Socrates, Plato, and the other instances are technically known as *individuals*. You may accordingly make the predication "Socrates is an individual." In a corresponding manner the intellect may leave out of consideration the specific differentia of man and have an object, "animal," that can be seen identified in reality equally well with horses and elephants and hundreds of

other species. So understood, the [scholastic] object is technically called a *genus*, and allows the predication "Animal is a genus." The higher genera are subject to the same process. They are the higher universals.

In all this reflexive activity the intellect is gazing not directly at the thing in its real existence, but at that thing as already objectified in various intentional existences. The view is now of a different kind from the first direct way of looking at the thing in the real world. It is now reflexion, and not direct cognition of a real thing. It is a second gaze at the thing, but now at the thing as found in a new intentional existence. This second or reflexive gaze [an "operation of the mind"—Eustachius (306n324)] at the thing in any of its various representations or objectivations is accordingly called the *second intention*. . . .

Two different types of predicate, therefore, may be applied to a sensible thing. On the one hand, there are the specific [man] and generic [animal] natures that characterize the thing wherever it is found, and the accidents that it has in the real world. These are predicates of the first intention. . . . On the other hand, there are predicates that of their nature apply just to a thing in its intentional being. These are characteristics that arise only when the thing receives cognitional being in the knower ["belonging to a thing known by virtue of its being known" (207) rather than by virtue of its real being]. . . .

Viewed in the first intention, Socrates and man and animal have but one [Thomistic] existential act. Viewed in the second intention, each has its own separate existential act. In real being, they are one. As [scholastic] objects of the second intention, on the other hand, they are diverse in being.

Each [scholastic] object of the second intention is accordingly a distinct "being of reason" (*ens rationis*). (*ECM* 239–41; bracketed explications mine)

The words "reflect," "reflection" or "reflexion," "reflexive" appear six times in this passage. It is the same sense of *reflexive* that Klein quotes from Eustachius on page 207. Again (as above), intellect is there looking both at itself and at the world: *at itself*, in its activity of classifying content known from looking *at the world* of real beings in their natural kinds or forms and (Thomistic) common natures. Scholastic reflexivity is restricted because it remains form-receptive.[28] This is the realism of Aristotelian-scholastic philosophy, often called naïve realism to distinguish it from scientific realism in twentieth-century philosophy of science. Greek philosophy generally is based on the premise of a harmony (despite serious distortions) between mind and world.[29] To repeat (it bears repeating), the contrast with Cartesian mind-world separation is salient.

V. Klein's Explanation of Cartesian Abstraction in Terms of Second Intention

At *GMTOA* 208, Klein extends the scholastic second-intention doctrine to a more radical, Cartesian sense of *cogito*-like reflexivity,[30] in which intellect—"*bare of any immediate reference to the world*" (201:–3)—exercises its unique power of "separating abstract entities" such as "mere multitude" (*solam multitudinem*), "multitudinousness as such," generic manyness:

> The intellect, when directed to the "idea" of a number as a "multitude of units" [say, points (201:–5)] . . . offered to it by the imagination . . . , turns [in the second intention, but] in conformity to its own [Cartesian] nature (*seiner eigenen Natur gemäss—GL*-II 221), toward its own "directedness," its own knowing. . . . Consequently it sees the multitude of units no longer "directly," . . . but "indirectly," "secondarily". . . .
>
> Its immediate "object"[31] is now its own conceiving [like

> the *cogito*] of that "multitude of units," that is, the "concept" (conceptus) of the number as such; nevertheless this multitude itself appears (*erscheint*—*GL*-II 221) as a "something," namely as *one* and therefore as an "ens," a "*being*." This is precisely what the abstraction which the intellect undertakes consists in: It transforms the multitude of the number into an apparently "independent" being, into an "*ens*," if only an "*ens rationis*." (208:1–20; the Halper epigraph refers to this text)

The mind's own conceiving is apprehended as something *one*. In the traditional understanding, a number (*arithmos*) is precisely not one, but many, such that the first number is two. Here, in *cogito*-like reflexivity, the Cartesian pure intellect "applies itself," not at all to the world of independently existing species, multitudes and magnitudes,[32] but only to the corporeal imagination. It reflects on its own cognition, something one, of the figure etched on the imagination (a multitude of units, say, a pattern of points). Pure intellect creates by its unique (non-scholastic) power of reflection-*cum*-abstraction the *concept* of the number as such, *sola multitudo*—mere or *just* multitude without regard to determinate amount. This concept is an *ens rationis* but not of scholastic type (based on determinate first intentions); rather it is *self-reflexive* and partly indeterminate. The mode of being of the object (the determinate multitude of units) is thus replaced by the mode of being of this distinctive concept. This concept is related to, but *not a likeness of* the object (a determinate amount of units)[33]; this is description 3 of symbolic conceptuality. To make this clearer, let us briefly review the most essential (*per se*) characteristics of discrete number (*arithmos*) and continuous magnitude (*megethos*) in the pre-modern understanding.

Consider the following: An *arithmos* is finitely divisible into indivisibles without boundaries that can touch. A *megethos* is infinitely divisible into parts having boundaries that can touch.

These statements express the differing *modes of being* of *arithmos* and *megethos*, which are thus like two different species within the genus of the quantified (*to poson*) whose common characteristics are divisibility and being subject to the equal and unequal.[34] Now, is our mentally *conceiving* of number or our *concept* of number (it makes no difference) finitely divisible into indivisible concept-units lacking boundaries that can touch? Is our *concept* of magnitude infinitely divisible into concept-parts having boundaries that can touch? No. However a mental entity may be understood as "stuff," as a material,[35] it is not divisible in the manner of number and magnitude (finitely vs. infinitely, having vs. lacking boundaries that can touch). Therefore, identifying number (*arithmos*)—specifically, identifying its mode of being, which, as just described, differs essentially from that of magnitude (*megethos*)—with the mode of being of the concept related to it, which registers no such difference, removes the distinction between discrete and continuous, leaving something one, intelligible (at least partially), but not imaginable, namely, *sola multitudo*, with which, at this point—having not yet arrived at *symbol*—we can do nothing except think it.

Recall the analogy between substance and number (as man and animal are to Socrates, so fiveness and manyness are to five units). As man and animal remain determinate (we can define them) so fiveness and manyness are determinate as the "form" of five units and as the defining characteristic of "finite divisibility to indivisibles lacking boundaries that can touch." On this account, the substance-number analogy leaves intact the discrete-continuous distinction[36] and, therefore, does not get to Descartes's *sola multitudo*. The Cartesian adjective *sola* ("mere" or, as I suggest, "just") would then be a sign of the shift in intentionality—the gaze of the intellect—from the mode of being of the objects (number and magnitude) to the mode of being of the one, self-reflexive concept related to those objects, *sola multitudo*, multitudinousness as such.[37]

Before taking up Klein on symbol (208:20–28), we have another significant question: pre-symbolic magnitude (*megethos*) is also per se divisible (though differently from number as described above); why is number (*arithmos*) privileged, as it were, in the movement via Cartesian abstraction to generic manyness? Is there not also a corresponding one *ens rationis* from the side of magnitude, namely, *magnitudinousness as such*? Both generic manyness and magnitudinousness are *entia rationis*, Cartesian second intentions. What is the content of generic manyness, *sola multitudo*, that lends it to symbolization in a way that generic magnitudinousness does not? I believe the answer is calculation: How can we calculate (add, multiply, subtract, divide) with continuous magnitudes unless they are conceived as numbers of units of measurement, either actual or potential? (123)[38]

In *GMTOA*, note 319, Klein mentions "figurality" itself ("*Figürlichkeit*" *überhaupt*; *GL*-II 216n189). This important note begins, "[h]ere it is important that the 'figures' appear as 'numbers' only through the '*mediating unit*' . . . and that the *unit* itself is understood as 'unit of measurement'." Klein makes clear (see especially 205) that in fact Descartes begins his explication of the new type of abstraction already from the standpoint of algebra. Indeed, "Descartes' thinking, as he himself points out in the *Regulae*, *presupposes* the fact of symbolic calculation, namely in the form of contemporary 'algebra'" (197). Descartes's objective is to make sense of it in such a way that the new mathematics, specifically, algebraic equations, can be justified as the language of physics. As to how symbolic-algebraic conceptuality historically developed, *GMTOA*, note 259, might pertain: there was "*a gradual change in the understanding of number*, whose ultimate roots lie too deep for discussion in this study" (277).

According to Klein, then, the Cartesian second intention, *sola multitudo*, can be abstracted by pure intellect from either points or lines etched on the corporeal imagination because Descartes preconceives lines as multitudes of units of measurement.

"This 'symbolic' character of Cartesian 'figures' first makes possible that mutual correspondence of 'lines' with [single] letters or 'ciphers' which obtains in Cartesian mathematics" (205). This concludes my account of two reflexivities.

At 208, line 21, Klein turns to description 4: symbols, and the paradoxical identification of a second intention with a first intention (more precisely, a second intend*ed* with a first intend*ed*), namely, the letter sign or line segment of the *Geometry*.

VI. Klein's Use of the Scholastic Intentionality Doctrine—Part II: Symbols

The next step in Klein (208:21) is the intellect-active employment of imagination, that is, the production of symbols, with which we can calculate.

> When now—and this is of crucial importance—the *ens rationis* as a "second intention" is grasped *with the aid of the imagination* in such a way that the intellect can, in turn, take it up as an object in the mode of a "first intention," we are dealing with a *symbol*, either with an "algebraic" letter-sign or with a "geometric" figure *as understood by Descartes*. This is the sense in which we spoke earlier of "symbolic abstraction."

The earlier passage to which Klein here refers concerns the role of symbolic abstraction in Viète's mathematical procedure:

> The intentional object of an "intentio secunda" is indicated by the letter-sign, namely, a concept that directly means a concept and not a being (*mit dem Buchstaben-Zeichen wird der intentionale Gegenstand einer "intentio secunda" bezeichnet, nämlich eines Begriffs, der selbst unmittelbar einen Begriff und nicht ein Seiendes meint*—*GL*-II 182). Furthermore—

> and this is the truly decisive turn—this general character of number or, what amounts to the same thing, this "general number" in all its indeterminateness, that is, in its merely possible determinateness, is accorded a certain independence which permits it to be the subject of "calculational" operations [whereby "intellect can . . . take it up as an object in the mode of a 'first intention'"]. This is achieved by adjoining the "rung" designations [indicating genera or dimensions], whose interconnection according to precise rules indicates the particular homogeneous field underlying each equation which is constructed. . . . *The species* [letter signs and associated genera or dimensions] *are in themselves symbolic formations—namely formations whose merely potential objectivity is understood as an actual objectivity.* They are, therefore, comprehensible only within the language *of symbolic formalism*. . . . Therewith the most important tool of mathematical natural science, the "*formula*," first becomes possible . . . but, above all, a new way of "understanding," inaccessible to ancient *episteme*, is thus opened up.[39] (174–75)

$E = mc^2$ is an emblematic example of a scientific formula.

Hopkins has provided detailed analysis of Klein on Viète, Descartes, and the peculiar first-and-second-intentional structure of algebraic symbols (see *RI* "Klein's Account of the Conceptual Presuppositions Belonging to Viète's Interpretation of Diophantine Logistic", *OLSM* Chapter 22–23 and §§200, 204, and 207). I put forward only the following two supplementary remarks.

First, it should be clear in light of Owens that the first-intentional character of the symbols (ciphers, letter signs, Cartesian line segments), despite their being sensible individuals, is not at all scholastic (realist): The content known in calculating with Viète's species, A, B or our x, v, t (distance, speed, time), and so on is not "present in" extramental beings "as first known" (Owens, *ECM* 237), that is, prior to learning the manipulations of algebra and dimensions of physics. In contrast, man and animal *are* first present in Socrates, but we cannot calculate with Socrates, man, animal. Therefore, it is calculability that provides A, B, x, v, t, etc., with first-intentional character as Klein understands it: "[I]t is obviously impossible to see 'numbers' in the isolated letter signs 'A' or 'B', except through the syntactical rules which Vieta states in the fourth chapter of the *Isogoge* [*sic*]" (176). It is in fact impossible to see *any* character in A, B, x, v, t (except the 1st, 2nd, 24th, 22nd, and 20th letters of the English alphabet) in isolation from our ability to "move them around" on the page, back and forth across the equal sign, e.g., x = vt, v = x/t, t = x/v, as we learn in freshman physics. Their calculability makes algebraic symbols first-intentional in Klein's (non-scholastic) sense. This at least is my claim.

Second, the passage from Klein on Viète, above, explicitly mentions *equation* ("each equation which is constructed"). This is a reminder that symbolic concept-formation comes into being historically in the arena of equations (in contrast to sentences or *logos*). An equation is a whole composed of multiple letter signs (variables and constants), signs for the operations (originally addition, subtraction, multiplication, division, taking of roots), and the equal sign between the right- and left-hand sides. In terms of Descartes's account in Rule 14, this means that pure intellect would need to separate from the corporeal imagination not just multitudinousness as an abstract entity, but also the binary operations, and the relation of equality. Equality is easy to accommodate: It falls in the Cartesian category of common simple natures (common to both the corporeal and the intellectual), which "can be known either by the pure intellect or by the intellect as it intuits the images of material things" (Rule 12, AT X 419; CSM I 45). The more challenging question is, how might Rule 14's symbolic abstraction work for the binary operations, and, very specifically, for Descartes's revolutionary "fourth-proportional" definition of multiplication in the opening pages of the *Geometry*? This important question is taken up by Romiti, whose work we are fortunate to have.[40]

Klein and his translator—to whom we in the English-speaking world owe a lot—provide an appropriate conclusion to the present endeavor:

> The whole complex of ontological problems which surrounds the ancient concept of number loses its object in the context of the symbolic conception, since there is no immediate occasion for questioning the mode of being of the "symbol" itself. (213)

"We tend to see and approach our lives and the world through a screen of concepts, techniques, and symbols." —Eva Brann[41]

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Two Reflexivities: Scholastic and Cartesian Second Intentions in Klein's *Greek Mathematical Thought and the Origin of Algebra* by Richard F. Hassing

1. The following abbreviations are used throughout: *GMTOA*: Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*, trans. Eva Brann (Cambridge, MA: M.I.T. Press, 1968). Numbers in parenthesis without an abbreviation refer to this text by the pages and (after a colon, when needed) lines; e.g., (206:5) means *GMTOA*, page 206, line 5; (198:–3) means page 198, third line from the bottom. *GL*-II: Jacob Klein, "Die griechische Logistik und die Entstehung der Algebra, Teil II," *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*, Abteiling B, Band 3, Heft 2 (Berlin: Julius Springer, Berlin: Julius Springer, 1936): 122–235. The German is cited only where there are small but significant differences between it and Brann's English translation, especially concerning the word 'object.' *WPNW*: Jacob Klein, "The World of Physics and the 'Natural' World," in *Lectures and Essays*, ed. Robert B. Williamson and Elliott Zuckerman (Annapolis, MD: St. John's College Press, 1985), 1–34. *MR*: Jacob Klein, "Modern Rationalism," in *Lectures and Essays*, 53–64. *OP*: Jacob Klein, "On Precision," in *Lectures and Essays*, 289–308. *OLSM*: Burt Hopkins, *The Origin of the Logic of Symbolic Mathematics: Edmund Husserl and Jacob Klein* (Bloomington, IN: Indiana University Press, 2011). *RI*: Burt Hopkins, "The Role of 'Intentionality' in Jacob Klein's Account of the Historicity of the Con-

cept of Number," in the present volume. *CM*: Andrew Romiti, *Cartesian Mind and Its Concept of Space: A Contribution to the Project of Jacob Klein*, Ph.D. dissertation, Catholic University, 2016. *ECM*: Joseph Owens, *An Elementary Christian Metaphysics* (Houston, TX: Center for Thomistic Studies, 1985). AT: *Oeuvres de Descartes*, ed. Charles Adam and Paul Tannery, 11 vols. (Paris: Vrin, 1996). Citations are to AT, volume and page number, and where full precision is required, a colon and line number; e.g., AT X 435:22. CSM: *The Philosophical Writings of Descartes*, trans. John Cottingham, Robert Stoothoff, and Dugald Murdoch, 2 vols. (Cambridge: Cambridge University Press, 1985).

2. Comment made at the 2nd Jacob Klein Conference, St. John's College, Annapolis, June 5, 2013. It refers to Klein, *GMTOA* 208, as discussed in Section V of this paper.

3. When we calculate, we combine two mathematical objects to make a third, e.g., by addition or multiplication: $2 + 3 = 5$, $2 \times 3 = 6$. We then generalize from the simple binary operations of the arithmetic of whole numbers to abstract rules ("axioms"), e.g., both addition and multiplication are *commutative*, $a + b = b + a$, $a \times b = b \times a$, multiplication is *distributive* over addition, $a \times (b + c) = a \times b + a \times c$, but addition is not distributive over multiplication, $a + (b \times c) \neq (a + b) \times (a + c)$. All objects of syntactically based axiomatic mathematics must be quantities, or qualities with quantitative aspects (like rotations around certain axes), not substances, in the sense that any two members of the relevant set are either equal or unequal, unlike substances. *The equation* is thus the basic object on which we work in mathematics and any science based on mathematics, e.g., physics. Algebraic letter signs in equations are analogous to syllables and words in sentences. By "the elevation of syntax" in equations (a phrase I owe to Robert Sokolowski) new operations and objects are created, e.g., imaginary numbers, whose squares are negative, and matrices, whose products in general do not commute. (Can we do the same thing with sentences and thereby make new realities?)

4. Significant texts, discussed in the following, also appear in Chapter 11 on Viète (174–75), and in Chapter 12A on Stevin (192).

5. See, on this score, *OLSM* 312–13, also *RI* (final paragraph of the Introduction): "[Klein] employs the scholastic distinction between first and second intentions in a non-scholastic way, in order to capture what he maintains is distinctive about the mathematical symbol's specific mode of being."

6. See Aristotle, *De Anima* 3.8, 431b20–432a4.

7. How to translate Klein's key term *symbolische Abstraktion*? Brann uses "symbol-generating abstraction" to avoid the implication that the abstrac-

tion by pure intellect is itself symbolic ("Preface" to *OLSM* xxvii, n. 3). Hopkins finds "symbolic abstraction" still appropriate; see *OLSM* 306–7n165. I use the latter; there are arguments pro and con for both translations.

8. *veram partem corporis* (AT X 414:20); *nihil aliud . . . quam verum corpus reale extensum et figuratum* (AT X 441:12–13); see also Descartes's *Optics*, Discourse 5 and 6. See note 13 for the problem with Descartes's insistence that the imagination is purely corporeal, part of the body-machine.

9. *se applicat, applicet se* (AT X 415:18 and 28); see also Descartes's Fifth Replies, AT VII 387:13.

10. *entia abstracta separandi* (AT X 444:23).

11. AT X 445:25

12. See also "indeterminate manyness" (201; "*bloße*" (*unbestimmte*) *Vielheit*—*GL*-II 211); "multiplicity in general," "'naked' multiplicity" (202; *Vielheit überhaupt*, "*bloße*" *Vielheit*—*GL*-II 211); "'mere manyness'," "multitudinousness as such" (207; *bloße Vielheit*, *Mengenhaftigkeit als solche*—*GL*-II 221).

13. Klein refers (more clearly in the German) to Descartes's imagination in its intellect-active function as the "power of imagination" or "imaginative power" (201, 202; *Einbildungskraft*, *imaginative Kraft*, also *Einbildungsvermögen*, *Vorstellungsvermögen*—*GL*-II 209, 210, 211). In Rules 12–14, Descartes consistently uses *imaginatio* or *phantasia*; *vis* [power] is reserved exclusively to the intellect (*vis cognoscens*—AT X 415:23). We do have "*vis . . . imaginandi*" at *Meditation* 2, AT VII 29:9–10, which seems to be a faculty of the *vis cognoscens* while the existence of all body is doubted. (Or is it? See Sixth Replies, AT VII 435:18–22, referring to the model of vision in *Optics*, Discourse 5 and 6, as tacitly presupposed for the argument of *Meditation* 2.) See Hopkins, *OLSM* 302–6 and 510n163, in support of Klein's usage. Salient for Klein's attribution of an imaginative *power* is simply that a true Cartesian body part (shaped, mobile extension) cannot have a power. Therefore, to the extent that Descartes's own account of intellect and imagination in Rule 14 in fact requires an imaginative power separate from the intellectual power (and I give Klein and Hopkins the benefit of the doubt), it reminds of Aristotle on powers of the soul, some of which (not imagination) are exercised through corresponding bodily organs (see, e.g., *De Anima* 2.1, 412a10), which organs thus cannot be mere Cartesian (inert) body parts. In other words, isn't Descartes putting a power in a power-less body part? The point is that, despite the value of Descartes's attempt to account for symbolic abstraction ("perhaps novel in the history of science"; 200), his model of pure intellect and corporeal imagination is not fully coherent. But this is hardly news: "How this mediation [by the imagination between the mind and the world] is to be understood

is, as is well known, *the* insoluble problem of Cartesian doctrine" (203). See *OLSM* §207, for a phenomenological account independent of psychophysics.

14. I thank Blaise Blain for this metaphor. Calculating with symbols is the analogy to feeding and petting. Accordingly, the disanalogy is that we don't need a *set* of animals for feeding and petting (one will do) whereas there must be a set of symbols, equal and unequal, and at least one binary operation by which to combine two elements of the set into a third. See note 3, above.

15. Symbols are imaginable (we see them on the paper in front of us) but they are not *images* of intelligible ideal originals or *signs* pointing to a number of pure intelligible units (such as Moγ, three units, in Diophantine notation) as were the numbers and diagrams of premodern mathematics. An algebraic symbol, unlike an image or sign, does not acquire its meaning from what it points to or images; rather (description 1, above), it acquires its meaning only within the symbol set as a whole with its rules of calculation. A symbol in physics has, in addition, a physical dimension (e.g., mass, length, area, volume, time, charge) and is thus related to a procedure of measurement whereby we can assign a number of kilograms to mass m, kilometers to distance x, seconds to time t, etc.

16. Is this the notorious Cartesian metaphysical (two-substance) dualism of *Discourse* 4 and *Meditation* 6? Romiti, *CM* 53–57, argues that Cartesian mind-world separation consists in a metaphysically neutral operational or functional dualism (in doing so, he follows Pamela Kraus, "*Mens Humana: Res Cogitans* and the Doctrine of Faculties in Descartes' *Meditations*," *International Studies in Philosophy* 18 [1986]: 4–13). More generally, the relation of mind and world in Descartes's science is a central theme of *CM*. Suffice to say that Descartes's conceiving of mind and world as separate in the *Rules* is not a scientific or philosophic discovery; it is a deliberate, willful decision, like his radical doubt. The intellectual simple natures of the *Rules* include cognition *and volition* (see Rules 6, 8 and 12; AT X 383, 399, 419). The thinking thing is also the willing thing.

17. The bracketed [not originally but] is Klein's. A question for Eustachius: If privation, for example, appertains to something in its own mode before any operations of the intellect, then why is a privation a being of reason? Perhaps in this sense: Consider the privation of sight in a cat. By nature (for the most part—see Aristotle's *Physics*, 196b11) cats have the power of vision, which is thus not a being of reason. There is no power of blindness. But blindness "appertains" to cats in that sight does belong to their feline mode of being. Privation is the lack of something in a being that, in its own mode, has the potency for it.

18. First intention is mentioned by Klein but only after second intention, and in connection with symbol, the goal of the account on page 208 of *GMTOA*: The letter sign is a sensible particular: we calculate with it, and thus deal with it as a "first intention" (*GL*-II 222). The word for "object" (208:23), *Gegenstand*, does not appear in the German.

19. Klein follows this order in *MR* (60–63) but he does not there discuss reflexivity.

20. See Aristotle, *Metaphysics* 7.13, 1039a4–6: one substance cannot be composed of many substances.

21. See, especially, Sixth Replies, point 9, AT VII 436–37 (CSM II 294–95), as well as Rule 8, "very much foreign"—AT X 398:13 (CSM I 31), Rule 12, AT X 412–17 (CSM I 40–43), *The World*, AT XI 3–15 (CSM I 81–85), *Principles of Philosophy* 2.23, AT VIIIA 52–53 (CSM I 232–33), *Passions of the Soul*, article 13, AT XI 338 (CSM I 333). From his early writings (*Rules* and *World*) to his late (*Principles* and *Passions*), Descartes rejects any notion of the irreducible wholeness of an organic being (e.g., a cat) in favor of universal corpuscular reductionism: the cat and the fire is a cloud of corpuscles (of three kinds) colliding in the plenum under the (hypothetical) three laws of nature. The mind can know figure, extension, motion because the corporeal imagination is extended, shaped, mobile.

22. See also 122 and 213.

23. More needs to be said. Didn't the ancient materialists (e.g., the atomists, Democritus and Lucretius) propose pre-symbolic-algebraic models? Yes, for example, the generation of a new offspring is, despite appearances, just a rearrangement in the void of swirling Democritean atoms. But the materialists did not have mathematical laws that enabled prediction and control of certain phenomena and thereby inspired belief in a comprehensive worldview entailing unlimited mastery of nature. See *GMTOA* 175 and 185; *WPNW* 30–34; *OP* 305 ("matching," as discussed in note 39 below).

24. See Aristotle, *Physics* 2.1–2.

25. Robert Sokolowski, in a private conversation, noted that the term *intention* here blurs the distinction between the intend*ing* activity and the target intend*ed*, and is a potential source of confusion. Where needed in the following, I try to state which is meant, the intend*ing* or the intend*ed*.

26. A precision: for Aristotle, a nature in the sense of *Physics* 2.1 belongs to the species of the informed substance, e.g., man, not to the genus, animal (see also *Metaphysics* 7.4, 1030a12–13). In the Thomistic account, animal as belonging to a man is called a "common nature" (see Owens, *ECM* 239n13).

27. More than one unit is needed for an *arithmos*, the first of which is accordingly two. The question of the unity of an *arithmos* is disputed

between Plato and Aristotle (see *GMTOA* 105–7), but any putative unity of an *arithmos* would presumably also be first-intentional, irreducible to but also inseparable from the determinate amount of units.

28. Aristotle, *De Anima* 3.8, 431b20–432a2. I believe this remains the case despite the reification of concepts characteristic of late (sixteenth and seventeenth century) scholasticism, about which Descartes complains in Rule 14, paragraphs 9–12 (AT X 443–45; CSM I 59–61). In *WPNW*, Klein says that the conceptuality (*Begrifflichkeit*) of the late scholastics was the same as that of the founders of the new science (6). However true this may be, the scholastic account of second intention and reflexivity in Eustachius (*GMTOA* 207) is not the same as what Klein finds (by adapting Eustachius) in Descartes (*GMTOA* 208), namely, *cogito*-like reflexivity, as discussed in Part V.

29. "[For the ancients,] though there is a clear distinction between mind and world, there is no separation between them, but rather mind is very emphatically the receiving of the world and nothing but that. . . . This is not just another philosophical theory but the very premise of their whole thought" (*MR* 58).

30. On Descartes's *cogito* ("I think therefore I am"; *locus classicus Disc.* 4, AT VI 32) as well as the distinctive *cogito*-like reflexivity, see the following from *Meditation* 2: "At last I have discovered it—thought; this alone is inseparable from me. I am, I exist. . . . [f]or as long as I am thinking. . . . I am, then, in the precise sense only a thing that thinks. . . . which is real and which truly exists" (AT VII 27; CSM II 18). "And here is the point, the perception I have of it [the wax] <or rather the act whereby it is perceived> is a case not of vision or touch or imagination—but of purely mental scrutiny. . . . the scrutiny of the mind alone" (AT VII 31–32, AT IXA 24; CSM II 21, and note 2 for the French).

31. The term *object* here can be either scholastic (it's in the intellect) or Cartesian (it's the target of the intellect's intending).

32. In the realist account, the unity of things as known (the unity of intentional beings) originates from, and remains rooted in, the unity of substances. "In general, those things are one in the highest degree if the thinking of their essence is indivisible . . . and of these the substances above all." Aristotle, *Metaphysics*, 1016b1–4. See also *Physics*, 192b33–35 and *Metaphysics*, 1052a30–34.

33. Aristotle, *De Anima* 3.8, 431b20–432a3, also *Nic. Ethics* 6.1, 1139a9–11. The concept here seems to be *of* the conceiving of a many without regard to how many. Any many can be combined (by +, −, ×, ⁄) with any other many to make a third many, given appropriate consideration to units and dimensions.

34. Aristotle, *Categories*, 4b26–33 & 5a1–5; *Physics* 5.3, 227a21–22 & 6.1; *Metaphysics* 5.13, 1020a7 & 12.10, 1075b28–30. Note that ratio—crucial for the development of modern mathematics—is not included in the category of the quantified but in that of relation (*pros ti*). Note also that the seven per se quantities (number, line, surface, solid, speech, time, place) are sorted by Aristotle according to three different divisions: discrete vs. continuous, having vs. not having relative position in the parts, having vs. not having permanence in the parts. In view of these additional divisions (beyond discrete and continuous), it is in fact unlikely that the category of quantity possesses the unity of a genus; it is too cut up and lacks ratio.

35. See Stevin's syllogism, *GMTOA* 191–92: "[Stevin] understands [*numerus* as 'a multitude . . . of units'] . . . as the 'material' of the thing to be defined, in the same sense in which one speaks of the material . . . of water or bread."

36. The problem, divide a square into two equal squares, would have a solution in magnitudes but not in numbers; due to incommensurability, there would be no univocal mathematical object, x, such that $x^2 = 2$.

37. On Viète's "Cartesian" self-reflexivity, see *OLSM* 521. According to Klein, the general theory of proportions meets "half-way" "Vieta's conception of a . . . '*general*' algebra which will be equally applicable to geometric magnitudes and numbers" (158). For example, proportions can be alternated for numbers, magnitudes, and times: if $a : b :: c : d$, then $a : c :: b : d$, in general. Proclus comments that "a certain common nature" of numbers, magnitudes, and times permits this (*GMTOA* 160). But, as Klein says, the Euclidean general theory of proportions only got halfway to a general object: proportions involving heterogeneous magnitudes or involving magnitudes and numbers cannot be alternated. In Greek mathematics, operations remain limited by objects. In contrast, in the species logistic of Viète's analytic art (see his *Introduction to the Analytic Art*, Chapter II, stipulations 14 and 15—*GMTOA* 323–34), proportions involving heterogeneous magnitudes can be resolved into equations with quotients and products of heterogeneous magnitudes, e.g., a magnitude with dimension cm^3 (volume) can be divided by a magnitude with dimension cm^2 (area) to yield a magnitude of dimension cm (length), in accordance with Viète's calculational axioms ("Precepts") and "law of homogeneity" (see *GMTOA* 324–38). Vietan magnitudes are thus numerical and dimensional, like quantities in physics. I submit that Euclidean magnitudes are non-numerical and dimensional (lines, not lengths; surfaces, not areas; solids, not volumes); Cartesian magnitudes are numerical and dimensionless, but can take any dimensions needed for problem-solving application. Prior to Viète, generality of method in the theory of propor-

tions did not imply generality of object: "on this above all Aristotle, *Posterior Analytics* A 24, 85a31–b3" (*GMTOA* 124).

38. Could we calculate with the genealogical tree or the triangle and square in Rule XIV (AT X 450–451, CSM I 64)?

"As 'arithmetician' Stevin no longer deals with numbers of units which are determinate in each case but with the *unlimited possibility of combining ciphers* according to definite rules of calculation'" (193). "[Stevin] thus once and for all fixes the ordinary understanding of the nature of number, for which being able 'to count' is tantamount to knowing how to handle 'ciphers'" (197).

39. On the final sentence of this quotation, see Klein, *OP* 304–6: equations and the models built on them make possible the "matching" of calculated numerical predictions and experimentally measured numerical results; it's as if modern number—ultimately the length of the digit string on which theory matches experiment—replaces *logos* for the understanding of the world. As Heidegger says, "[P]hysics . . . will never be able to renounce this one thing: that nature reports itself in some way or other that is identifiable through calculation and that it remains orderable as a system of information [e.g., 0's and 1's]" (*The Question concerning Technology*, trans. William Lovitt [New York: Harper and Row, 1977], 23). For the link between the calculability of symbols and their first-intentional status see the paragraphs immediately following.

40. See *CM* 118–33.

41. Eva Brann, "Immediacy: The Ways of Humanity," *The Imaginative Conservative*, November 1, 2014 (originally given as a lecture, St. John's College, Santa Fe, October 2014).